Chapter 33

Coulomb Functions

I. J. Thompson[1]

Notation 742
33.1 Special Notation . . . 742

Variables ρ, η 742
33.2 Definitions and Basic Properties . . . 742
33.3 Graphics . . . 743
33.4 Recurrence Relations and Derivatives . . . 744
33.5 Limiting Forms for Small ρ, Small $|\eta|$, or Large ℓ . . . 744
33.6 Power-Series Expansions in ρ . . . 745
33.7 Integral Representations . . . 745
33.8 Continued Fractions . . . 745
33.9 Expansions in Series of Bessel Functions . . . 745
33.10 Limiting Forms for Large ρ or Large $|\eta|$ . . . 746
33.11 Asymptotic Expansions for Large ρ . . . 747
33.12 Asymptotic Expansions for Large η . . . 747
33.13 Complex Variable and Parameters . . . 748

Variables r, ϵ 748
33.14 Definitions and Basic Properties . . . 748
33.15 Graphics . . . 749
33.16 Connection Formulas . . . 751
33.17 Recurrence Relations and Derivatives . . . 752
33.18 Limiting Forms for Large ℓ . . . 752
33.19 Power-Series Expansions in r . . . 752
33.20 Expansions for Small $|\epsilon|$ . . . 752
33.21 Asymptotic Approximations for Large $|r|$ . . . 753

Physical Applications 753
33.22 Particle Scattering and Atomic and Molecular Spectra . . . 753

Computation 755
33.23 Methods of Computation . . . 755
33.24 Tables . . . 755
33.25 Approximations . . . 756
33.26 Software . . . 756

References 756

[1]Lawrence Livermore National Laboratory, Livermore, California.
Acknowledgments: This chapter is based in part on Abramowitz and Stegun (1964, Chapter 14) by M. Abramowitz.

Notation

33.1 Special Notation

(For other notation see pp. xiv and 873.)

k, ℓ	nonnegative integers.
r, x	real variables.
ρ	nonnegative real variable.
ϵ, η	real parameters.
$\psi(x)$	logarithmic derivative of $\Gamma(x)$; see §5.2(i).
$\delta(x)$	Dirac delta; see §1.17.
primes	derivatives with respect to the variable.

The main functions treated in this chapter are first the Coulomb radial functions $F_\ell(\eta,\rho)$, $G_\ell(\eta,\rho)$, $H^{\pm}_\ell(\eta,\rho)$ (Sommerfeld (1928)), which are used in the case of repulsive Coulomb interactions, and secondly the functions $f(\epsilon,\ell;r)$, $h(\epsilon,\ell;r)$, $s(\epsilon,\ell;r)$, $c(\epsilon,\ell;r)$ (Seaton (1982, 2002)), which are used in the case of attractive Coulomb interactions.

Alternative Notations

Curtis (1964a): $P_\ell(\epsilon,r) = (2\ell+1)!\, f(\epsilon,\ell;r)/2^{\ell+1}$, $Q_\ell(\epsilon,r) = -(2\ell+1)!\, h(\epsilon,\ell;r)/(2^{\ell+1}A(\epsilon,\ell))$.

Greene *et al.* (1979): $f^{(0)}(\epsilon,\ell;r) = f(\epsilon,\ell;r)$, $f(\epsilon,\ell;r) = s(\epsilon,\ell;r)$, $g(\epsilon,\ell;r) = c(\epsilon,\ell;r)$.

Variables ρ, η

33.2 Definitions and Basic Properties

33.2(i) Coulomb Wave Equation

33.2.1
$$\frac{d^2w}{d\rho^2} + \left(1 - \frac{2\eta}{\rho} - \frac{\ell(\ell+1)}{\rho^2}\right) w = 0, \quad \ell = 0,1,2,\dots.$$

This differential equation has a regular singularity at $\rho = 0$ with indices $\ell+1$ and $-\ell$, and an irregular singularity of rank 1 at $\rho = \infty$ (§§2.7(i), 2.7(ii)). There are two turning points, that is, points at which $d^2w/d\rho^2 = 0$ (§2.8(i)). The outer one is given by

33.2.2
$$\rho_{\mathrm{tp}}(\eta,\ell) = \eta + (\eta^2 + \ell(\ell+1))^{1/2}.$$

33.2(ii) Regular Solution $F_\ell(\eta,\rho)$

The function $F_\ell(\eta,\rho)$ is recessive (§2.7(iii)) at $\rho = 0$, and is defined by

33.2.3
$$F_\ell(\eta,\rho) = C_\ell(\eta) 2^{-\ell-1} (\mp i)^{\ell+1} M_{\pm i\eta, \ell+\frac{1}{2}}(\pm 2i\rho),$$

or equivalently

33.2.4
$$F_\ell(\eta,\rho) = C_\ell(\eta)\rho^{\ell+1} e^{\mp i\rho} M(\ell+1\mp i\eta, 2\ell+2, \pm 2i\rho),$$

where $M_{\kappa,\mu}(z)$ and $M(a,b,z)$ are defined in §§13.14(i) and 13.2(i), and

33.2.5
$$C_\ell(\eta) = \frac{2^\ell e^{-\pi\eta/2} |\Gamma(\ell+1+i\eta)|}{(2\ell+1)!}.$$

The choice of ambiguous signs in (33.2.3) and (33.2.4) is immaterial, provided that either all upper signs are taken, or all lower signs are taken. This is a consequence of Kummer's transformation (§13.2(vii)).

$F_\ell(\eta,\rho)$ is a real and analytic function of ρ on the open interval $0 < \rho < \infty$, and also an analytic function of η when $-\infty < \eta < \infty$.

The *normalizing constant* $C_\ell(\eta)$ is always positive, and has the alternative form

33.2.6
$$C_\ell(\eta) = \frac{2^\ell \left((2\pi\eta/(e^{2\pi\eta}-1)) \prod_{k=1}^{\ell}(\eta^2+k^2)\right)^{1/2}}{(2\ell+1)!}.$$

33.2(iii) Irregular Solutions $G_\ell(\eta,\rho), H^{\pm}_\ell(\eta,\rho)$

The functions $H^{\pm}_\ell(\eta,\rho)$ are defined by

33.2.7
$$H^{\pm}_\ell(\eta,\rho) = (\mp i)^\ell e^{(\pi\eta/2)\pm i\,\sigma_\ell(\eta)} W_{\mp i\eta,\ell+\frac{1}{2}}(\mp 2i\rho),$$

or equivalently

33.2.8
$$H^{\pm}_\ell(\eta,\rho) = e^{\pm i\,\theta_\ell(\eta,\rho)} (\mp 2i\rho)^{\ell+1\pm i\eta} U(\ell+1\pm i\eta, 2\ell+2, \mp 2i\rho),$$

where $W_{\kappa,\mu}(z)$, $U(a,b,z)$ are defined in §§13.14(i) and 13.2(i),

33.2.9
$$\theta_\ell(\eta,\rho) = \rho - \eta\ln(2\rho) - \tfrac{1}{2}\ell\pi + \sigma_\ell(\eta),$$

and

33.2.10
$$\sigma_\ell(\eta) = \operatorname{ph}\Gamma(\ell+1+i\eta),$$

the branch of the phase in (33.2.10) being zero when $\eta = 0$ and continuous elsewhere. $\sigma_\ell(\eta)$ is the *Coulomb phase shift*.

$H^{+}_\ell(\eta,\rho)$ and $H^{-}_\ell(\eta,\rho)$ are complex conjugates, and their real and imaginary parts are given by

33.2.11
$$H^{+}_\ell(\eta,\rho) = G_\ell(\eta,\rho) + i\,F_\ell(\eta,\rho),$$
$$H^{-}_\ell(\eta,\rho) = G_\ell(\eta,\rho) - i\,F_\ell(\eta,\rho).$$

As in the case of $F_\ell(\eta,\rho)$, the solutions $H^{\pm}_\ell(\eta,\rho)$ and $G_\ell(\eta,\rho)$ are analytic functions of ρ when $0 < \rho < \infty$. Also, $e^{\mp i\,\sigma_\ell(\eta)} H^{\pm}_\ell(\eta,\rho)$ are analytic functions of η when $-\infty < \eta < \infty$.

33.2(iv) Wronskians and Cross-Product

With arguments η, ρ suppressed,

33.2.12
$$\mathscr{W}\{G_\ell, F_\ell\} = \mathscr{W}\{H^{\pm}_\ell, F_\ell\} = 1.$$

33.2.13
$$F_{\ell-1}G_\ell - F_\ell G_{\ell-1} = \ell/(\ell^2+\eta^2)^{1/2}, \qquad \ell \geq 1.$$

33.3 Graphics

33.3(i) Line Graphs of the Coulomb Radial Functions $F_\ell(\eta,\rho)$ and $G_\ell(\eta,\rho)$

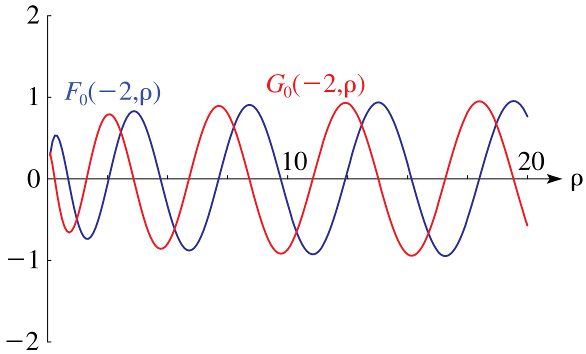

Figure 33.3.1: $F_\ell(\eta,\rho)$, $G_\ell(\eta,\rho)$ with $\ell=0$, $\eta=-2$.

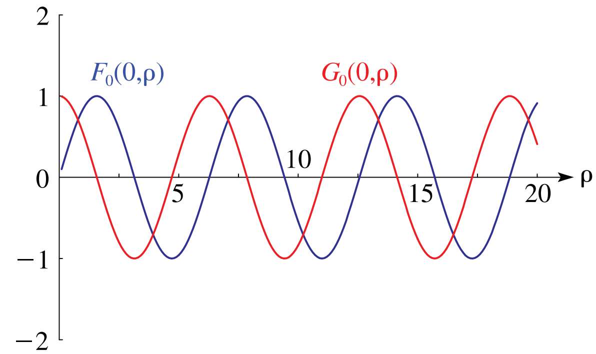

Figure 33.3.2: $F_\ell(\eta,\rho)$, $G_\ell(\eta,\rho)$ with $\ell=0$, $\eta=0$.

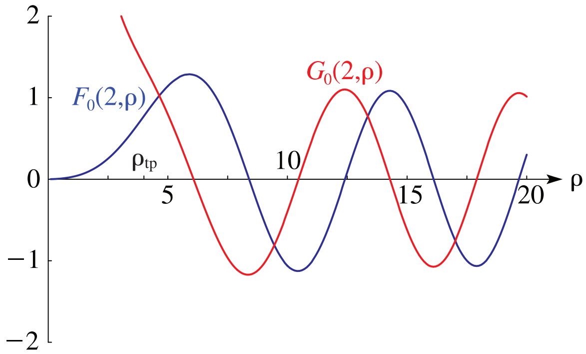

Figure 33.3.3: $F_\ell(\eta,\rho)$, $G_\ell(\eta,\rho)$ with $\ell=0$, $\eta=2$. The turning point is at $\rho_{\mathrm{tp}}(2,0)=4$.

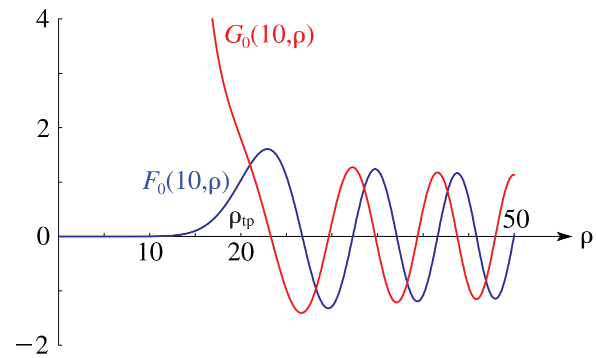

Figure 33.3.4: $F_\ell(\eta,\rho)$, $G_\ell(\eta,\rho)$ with $\ell=0$, $\eta=10$. The turning point is at $\rho_{\mathrm{tp}}(10,0)=20$.

In Figures 33.3.5 and 33.3.6

33.3.1 $$M_\ell(\eta,\rho)=(F_\ell^2(\eta,\rho)+G_\ell^2(\eta,\rho))^{1/2}=\left|H_\ell^{\pm}(\eta,\rho)\right|.$$

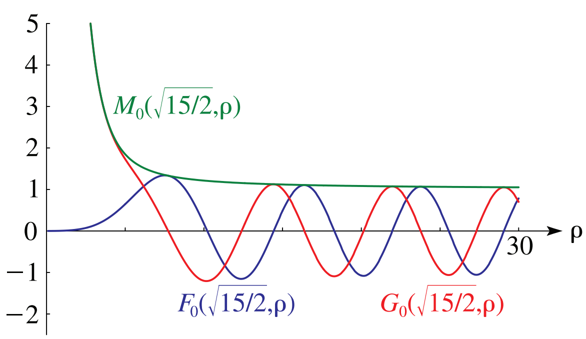

Figure 33.3.5: $F_\ell(\eta,\rho)$, $G_\ell(\eta,\rho)$, and $M_\ell(\eta,\rho)$ with $\ell=0$, $\eta=\sqrt{15/2}$. The turning point is at $\rho_{\mathrm{tp}}\left(\sqrt{15/2},0\right)=\sqrt{30}=5.47\dots$.

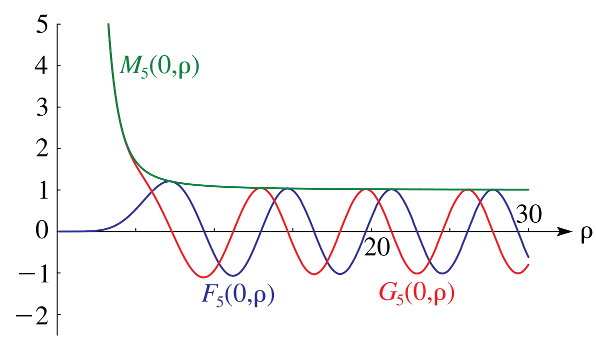

Figure 33.3.6: $F_\ell(\eta,\rho)$, $G_\ell(\eta,\rho)$, and $M_\ell(\eta,\rho)$ with $\ell=5$, $\eta=0$. The turning point is at $\rho_{\mathrm{tp}}(0,5)=\sqrt{30}$ (as in Figure 33.3.5).

33.3(ii) Surfaces of the Coulomb Radial Functions $F_0(\eta,\rho)$ and $G_0(\eta,\rho)$

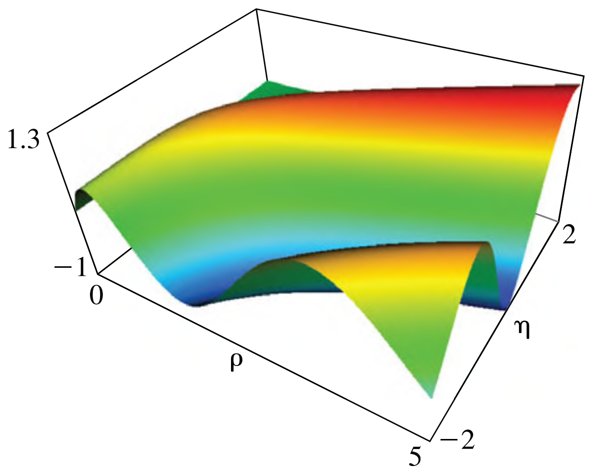

Figure 33.3.7: $F_0(\eta,\rho)$, $-2\le\eta\le 2$, $0\le\rho\le 5$.

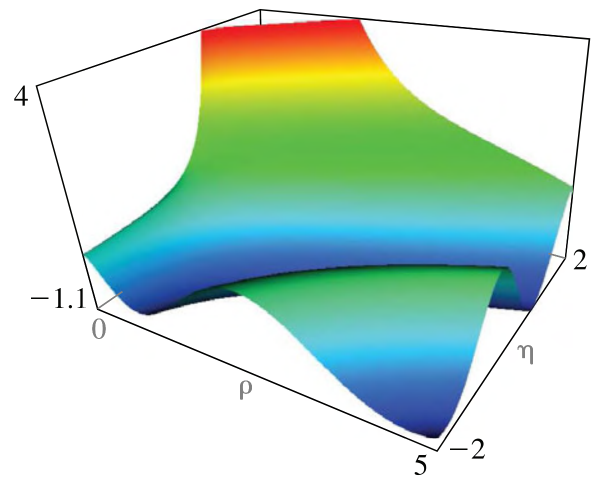

Figure 33.3.8: $G_0(\eta,\rho)$, $-2\le\eta\le 2$, $0<\rho\le 5$.

33.4 Recurrence Relations and Derivatives

For $\ell=1,2,3,\dots$, let

33.4.1 $$R_\ell=\sqrt{1+\frac{\eta^2}{\ell^2}},\quad S_\ell=\frac{\ell}{\rho}+\frac{\eta}{\ell},\quad T_\ell=S_\ell+S_{\ell+1}.$$

Then, with X_ℓ denoting any of $F_\ell(\eta,\rho)$, $G_\ell(\eta,\rho)$, or $H^{\pm}_\ell(\eta,\rho)$,

33.4.2 $$R_\ell X_{\ell-1}-T_\ell X_\ell+R_{\ell+1}X_{\ell+1}=0,\qquad \ell\ge 1,$$

33.4.3 $$X'_\ell=R_\ell X_{\ell-1}-S_\ell X_\ell,\qquad \ell\ge 1,$$

33.4.4 $$X'_\ell=S_{\ell+1}X_\ell-R_{\ell+1}X_{\ell+1},\qquad \ell\ge 0.$$

33.5 Limiting Forms for Small ρ, Small $|\eta|$, or Large ℓ

33.5(i) Small ρ

As $\rho\to 0$ with η fixed,

33.5.1 $$F_\ell(\eta,\rho)\sim C_\ell(\eta)\rho^{\ell+1},\quad F'_\ell(\eta,\rho)\sim(\ell+1)\,C_\ell(\eta)\rho^{\ell}.$$

33.5.2 $$G_\ell(\eta,\rho)\sim\frac{\rho^{-\ell}}{(2\ell+1)\,C_\ell(\eta)},\qquad \ell=0,1,2,\dots,$$
$$G'_\ell(\eta,\rho)\sim-\frac{\ell\rho^{-\ell-1}}{(2\ell+1)\,C_\ell(\eta)},\qquad \ell=1,2,3,\dots.$$

33.5(ii) $\eta=0$

33.5.3 $$F_\ell(0,\rho)=\rho\,\mathsf{j}_\ell(\rho),\quad G_\ell(0,\rho)=-\rho\,\mathsf{y}_\ell(\rho).$$

Equivalently,

33.5.4 $$F_\ell(0,\rho)=(\pi\rho/2)^{1/2}\,J_{\ell+\frac12}(\rho),$$
$$G_\ell(0,\rho)=-(\pi\rho/2)^{1/2}\,Y_{\ell+\frac12}(\rho).$$

For the functions j, y, J, Y see §§10.47(ii), 10.2(ii).

33.5.5 $$F_0(0,\rho)=\sin\rho,\quad G_0(0,\rho)=\cos\rho,\quad H^{\pm}_0(0,\rho)=e^{\pm i\rho}.$$

33.5.6 $$C_\ell(0)=\frac{2^\ell\ell!}{(2\ell+1)!}=\frac{1}{(2\ell+1)!!}.$$

33.5(iii) Small $|\eta|$

33.5.7 $$\sigma_0(\eta)\sim-\gamma\eta,\qquad \eta\to 0,$$

where γ is Euler's constant (§5.2(ii)).

33.5(iv) Large ℓ

As $\ell\to\infty$ with η and ρ ($\ne 0$) fixed,

33.5.8 $$F_\ell(\eta,\rho)\sim C_\ell(\eta)\rho^{\ell+1},\quad G_\ell(\eta,\rho)\sim\frac{\rho^{-\ell}}{(2\ell+1)\,C_\ell(\eta)},$$

33.5.9 $$C_\ell(\eta)\sim\frac{e^{-\pi\eta/2}}{(2\ell+1)!!}\sim e^{-\pi\eta/2}\frac{e^\ell}{\sqrt2(2\ell)^{\ell+1}}.$$

33.6 Power-Series Expansions in ρ

33.6.1
$$F_\ell(\eta,\rho) = C_\ell(\eta) \sum_{k=\ell+1}^{\infty} A_k^\ell(\eta)\rho^k,$$

33.6.2
$$F'_\ell(\eta,\rho) = C_\ell(\eta) \sum_{k=\ell+1}^{\infty} kA_k^\ell(\eta)\rho^{k-1},$$

where $A_{\ell+1}^\ell = 1$, $A_{\ell+2}^\ell = \eta/(\ell+1)$, and

33.6.3
$$(k+\ell)(k-\ell-1)A_k^\ell = 2\eta A_{k-1}^\ell - A_{k-2}^\ell, \quad k = \ell+3, \ell+4, \dots,$$

or in terms of the hypergeometric function (§§15.1, 15.2(i)),

33.6.4
$$A_k^\ell(\eta) = \frac{(-i)^{k-\ell-1}}{(k-\ell-1)!}\, {}_2F_1(\ell+1-k, \ell+1-i\eta; 2\ell+2; 2).$$

33.6.5
$$H_\ell^\pm(\eta,\rho) = \frac{e^{\pm i\,\theta_\ell(\eta,\rho)}}{(2\ell+1)!\,\Gamma(-\ell+i\eta)} \left(\sum_{k=0}^{\infty} \frac{(a)_k}{(2\ell+2)_k k!} (\mp 2i\rho)^{a+k} \left(\ln(\mp 2i\rho) + \psi(a+k) - \psi(1+k) - \psi(2\ell+2+k)\right) - \sum_{k=1}^{2\ell+1} \frac{(2\ell+1)!(k-1)!}{(2\ell+1-k)!(1-a)_k} (\mp 2i\rho)^{a-k} \right),$$

where $a = 1+\ell \pm i\eta$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$ (§5.2(i)).

The series (33.6.1), (33.6.2), and (33.6.5) converge for all finite values of ρ. Corresponding expansions for $H_\ell^{\pm\prime}(\eta,\rho)$ can be obtained by combining (33.6.5) with (33.4.3) or (33.4.4).

33.7 Integral Representations

33.7.1
$$F_\ell(\eta,\rho) = \frac{\rho^{\ell+1} 2^\ell e^{i\rho-(\pi\eta/2)}}{|\Gamma(\ell+1+i\eta)|} \int_0^1 e^{-2i\rho t} t^{\ell+i\eta}(1-t)^{\ell-i\eta}\,dt,$$

33.7.2
$$H_\ell^-(\eta,\rho) = \frac{e^{-i\rho}\rho^{-\ell}}{(2\ell+1)!\,C_\ell(\eta)} \int_0^\infty e^{-t} t^{\ell-i\eta}(t+2i\rho)^{\ell+i\eta}\,dt,$$

33.7.3
$$H_\ell^-(\eta,\rho) = \frac{-ie^{-\pi\eta}\rho^{\ell+1}}{(2\ell+1)!\,C_\ell(\eta)} \int_0^\infty \left(\frac{\exp(-i(\rho\tanh t - 2\eta t))}{(\cosh t)^{2\ell+2}} + i(1+t^2)^\ell \exp(-\rho t + 2\eta \arctan t) \right) dt,$$

33.7.4
$$H_\ell^+(\eta,\rho) = \frac{ie^{-\pi\eta}\rho^{\ell+1}}{(2\ell+1)!\,C_\ell(\eta)} \int_{-1}^{-i\infty} e^{-i\rho t}(1-t)^{\ell-i\eta}(1+t)^{\ell+i\eta}\,dt.$$

Noninteger powers in (33.7.1)–(33.7.4) and the arctangent assume their principal values (§§4.2(i), 4.2(iv), 4.23(ii)).

33.8 Continued Fractions

With arguments η, ρ suppressed,

33.8.1
$$\frac{F'_\ell}{F_\ell} = S_{\ell+1} - \frac{R_{\ell+1}^2}{T_{\ell+1}-}\,\frac{R_{\ell+2}^2}{T_{\ell+2}-}\,\cdots.$$

For R, S, and T see (33.4.1).

33.8.2
$$\frac{H_\ell^{\pm\prime}}{H_\ell^\pm} = c \pm \frac{i}{\rho}\,\frac{ab}{2(\rho-\eta\pm i)+}\,\frac{(a+1)(b+1)}{2(\rho-\eta\pm 2i)+}\,\cdots,$$

where

33.8.3
$$a = 1+\ell\pm i\eta, \quad b = -\ell \pm i\eta, \quad c = \pm i(1-(\eta/\rho)).$$

The continued fraction (33.8.1) converges for all finite values of ρ, and (33.8.2) converges for all $\rho \neq 0$.

If we denote $u = F'_\ell/F_\ell$ and $p + iq = H_\ell^{+\prime}\big/H_\ell^+$, then

33.8.4
$$F_\ell = \pm(q^{-1}(u-p)^2+q)^{-1/2}, \quad F'_\ell = u\,F_\ell,$$

33.8.5
$$G_\ell = q^{-1}(u-p)\,F_\ell, \quad G'_\ell = q^{-1}(up-p^2-q^2)\,F_\ell.$$

The ambiguous sign in (33.8.4) has to agree with that of the final denominator in (33.8.1) when the continued fraction has converged to the required precision. For proofs and further information see Barnett *et al.* (1974) and Barnett (1996).

33.9 Expansions in Series of Bessel Functions

33.9(i) Spherical Bessel Functions

33.9.1
$$F_\ell(\eta,\rho) = \rho\sum_{k=0}^{\infty} a_k\,\mathsf{j}_{\ell+k}(\rho),$$

where the function j is as in §10.47(ii), $a_{-1} = 0$, $a_0 = (2\ell+1)!!\, C_\ell(\eta)$, and

33.9.2
$$\frac{k(k+2\ell+1)}{2k+2\ell+1} a_k - 2\eta a_{k-1} + \frac{(k-2)(k+2\ell-1)}{2k+2\ell-3} a_{k-2} = 0, \quad k = 1, 2, \ldots.$$

The series (33.9.1) converges for all finite values of η and ρ.

33.9(ii) Bessel Functions and Modified Bessel Functions

In this subsection the functions J, I, and K are as in §§10.2(ii) and 10.25(ii).

With $t = 2|\eta|\rho$,

33.9.3
$$F_\ell(\eta,\rho) = C_\ell(\eta) \frac{(2\ell+1)!}{(2\eta)^{2\ell+1}} \rho^{-\ell} \sum_{k=2\ell+1}^{\infty} b_k t^{k/2} I_k\left(2\sqrt{t}\right), \quad \eta > 0,$$

33.9.4
$$F_\ell(\eta,\rho) = C_\ell(\eta) \frac{(2\ell+1)!}{(2|\eta|)^{2\ell+1}} \rho^{-\ell} \sum_{k=2\ell+1}^{\infty} b_k t^{k/2} J_k\left(2\sqrt{t}\right), \quad \eta < 0.$$

Here $b_{2\ell} = b_{2\ell+2} = 0$, $b_{2\ell+1} = 1$, and

33.9.5
$$4\eta^2(k-2\ell)b_{k+1} + kb_{k-1} + b_{k-2} = 0, \quad k = 2\ell+2, 2\ell+3, \ldots.$$

The series (33.9.3) and (33.9.4) converge for all finite positive values of $|\eta|$ and ρ.

Next, as $\eta \to +\infty$ with $\rho\ (>0)$ fixed,

33.9.6
$$G_\ell(\eta,\rho) \sim \frac{\rho^{-\ell}}{(\ell+\frac{1}{2})\lambda_\ell(\eta)\, C_\ell(\eta)} \sum_{k=2\ell+1}^{\infty} (-1)^k b_k t^{k/2} K_k\left(2\sqrt{t}\right),$$

where

33.9.7
$$\lambda_\ell(\eta) \sim \sum_{k=2\ell+1}^{\infty} (-1)^k (k-1)! b_k.$$

For other asymptotic expansions of $G_\ell(\eta,\rho)$ see Fröberg (1955, §8) and Humblet (1985).

33.10 Limiting Forms for Large ρ or Large $|\eta|$

33.10(i) Large ρ

As $\rho \to \infty$ with η fixed,

33.10.1
$$F_\ell(\eta,\rho) = \sin(\theta_\ell(\eta,\rho)) + o(1),$$
$$G_\ell(\eta,\rho) = \cos(\theta_\ell(\eta,\rho)) + o(1),$$

33.10.2
$$H^{\pm}_\ell(\eta,\rho) \sim \exp(\pm i\, \theta_\ell(\eta,\rho)),$$

where $\theta_\ell(\eta,\rho)$ is defined by (33.2.9).

33.10(ii) Large Positive η

As $\eta \to \infty$ with ρ fixed,

33.10.3
$$F_\ell(\eta,\rho) \sim \frac{(2\ell+1)!\, C_\ell(\eta)}{(2\eta)^{\ell+1}} (2\eta\rho)^{1/2}\, I_{2\ell+1}\left((8\eta\rho)^{1/2}\right),$$
$$G_\ell(\eta,\rho) \sim \frac{2(2\eta)^\ell}{(2\ell+1)!\, C_\ell(\eta)} (2\eta\rho)^{1/2}\, K_{2\ell+1}\left((8\eta\rho)^{1/2}\right).$$

In particular, for $\ell = 0$,

33.10.4
$$F_0(\eta,\rho) \sim e^{-\pi\eta} (\pi\rho)^{1/2}\, I_1\left((8\eta\rho)^{1/2}\right),$$
$$G_0(\eta,\rho) \sim 2e^{\pi\eta} \left(\rho/\pi\right)^{1/2} K_1\left((8\eta\rho)^{1/2}\right),$$

33.10.5
$$F'_0(\eta,\rho) \sim e^{-\pi\eta} (2\pi\eta)^{1/2}\, I_0\left((8\eta\rho)^{1/2}\right),$$
$$G'_0(\eta,\rho) \sim -2e^{\pi\eta} \left(2\eta/\pi\right)^{1/2} K_0\left((8\eta\rho)^{1/2}\right).$$

Also,

33.10.6
$$\sigma_0(\eta) = \eta(\ln\eta - 1) + \tfrac{1}{4}\pi + o(1),$$
$$C_0(\eta) \sim (2\pi\eta)^{1/2} e^{-\pi\eta}.$$

33.10(iii) Large Negative η

As $\eta \to -\infty$ with ρ fixed,

33.10.7
$$F_\ell(\eta,\rho) = \frac{(2\ell+1)!\, C_\ell(\eta)}{(-2\eta)^{\ell+1}} \left((-2\eta\rho)^{1/2} \times J_{2\ell+1}\left((-8\eta\rho)^{1/2}\right) + o\left(|\eta|^{1/4}\right)\right),$$
$$G_\ell(\eta,\rho) = -\frac{\pi(-2\eta)^\ell}{(2\ell+1)!\, C_\ell(\eta)} \left((-2\eta\rho)^{1/2} \times Y_{2\ell+1}\left((-8\eta\rho)^{1/2}\right) + o\left(|\eta|^{1/4}\right)\right).$$

In particular, for $\ell = 0$,

33.10.8
$$F_0(\eta,\rho) = (\pi\rho)^{1/2}\, J_1\left((-8\eta\rho)^{1/2}\right) + o\left(|\eta|^{-1/4}\right),$$
$$G_0(\eta,\rho) = -(\pi\rho)^{1/2}\, Y_1\left((-8\eta\rho)^{1/2}\right) + o\left(|\eta|^{-1/4}\right).$$

33.10.9
$$F'_0(\eta,\rho) = (-2\pi\eta)^{1/2}\, J_0\left((-8\eta\rho)^{1/2}\right) + o\left(|\eta|^{1/4}\right),$$
$$G'_0(\eta,\rho) = -(-2\pi\eta)^{1/2}\, Y_0\left((-8\eta\rho)^{1/2}\right) + o\left(|\eta|^{1/4}\right).$$

Also,

33.10.10
$$\sigma_0(\eta) = \eta(\ln(-\eta) - 1) - \tfrac{1}{4}\pi + o(1), \quad C_0(\eta) \sim (-2\pi\eta)^{1/2}.$$

33.11 Asymptotic Expansions for Large ρ

For large ρ, with ℓ and η fixed,

33.11.1 $$H_\ell^{\pm}(\eta,\rho) = e^{\pm i\,\theta_\ell(\eta,\rho)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(\mp 2i\rho)^k},$$

where $\theta_\ell(\eta,\rho)$ is defined by (33.2.9), and a and b are defined by (33.8.3).

With arguments (η,ρ) suppressed, an equivalent formulation is given by

33.11.2 $$F_\ell = g\cos\theta_\ell + f\sin\theta_\ell, \quad G_\ell = f\cos\theta_\ell - g\sin\theta_\ell,$$

33.11.3 $$F'_\ell = \widehat{g}\cos\theta_\ell + \widehat{f}\sin\theta_\ell, \quad G'_\ell = \widehat{f}\cos\theta_\ell - \widehat{g}\sin\theta_\ell,$$

33.11.4 $$H_\ell^{\pm} = e^{\pm i\,\theta_\ell}(f \pm ig),$$

where

33.11.5 $$f \sim \sum_{k=0}^{\infty} f_k, \quad g \sim \sum_{k=0}^{\infty} g_k,$$

33.11.6 $$\widehat{f} \sim \sum_{k=0}^{\infty} \widehat{f}_k, \quad \widehat{g} \sim \sum_{k=0}^{\infty} \widehat{g}_k,$$

33.11.7 $$g\widehat{f} - f\widehat{g} = 1.$$

Here $f_0 = 1$, $g_0 = 0$, $\widehat{f}_0 = 0$, $\widehat{g}_0 = 1 - (\eta/\rho)$, and for $k = 0, 1, 2, \dots$,

33.11.8 $$\begin{aligned} f_{k+1} &= \lambda_k f_k - \mu_k g_k, \\ g_{k+1} &= \lambda_k g_k + \mu_k f_k, \\ \widehat{f}_{k+1} &= \lambda_k \widehat{f}_k - \mu_k \widehat{g}_k - (f_{k+1}/\rho), \\ \widehat{g}_{k+1} &= \lambda_k \widehat{g}_k + \mu_k \widehat{f}_k - (g_{k+1}/\rho), \end{aligned}$$

where

33.11.9 $$\lambda_k = \frac{(2k+1)\eta}{(2k+2)\rho}, \quad \mu_k = \frac{\ell(\ell+1) - k(k+1) + \eta^2}{(2k+2)\rho}.$$

33.12 Asymptotic Expansions for Large η

33.12(i) Transition Region

When $\ell = 0$ and $\eta > 0$, the outer turning point is given by $\rho_{\rm tp}(\eta, 0) = 2\eta$; compare (33.2.2). Define

33.12.1 $$x = (2\eta - \rho)/(2\eta)^{1/3}, \quad \mu = (2\eta)^{2/3}.$$

Then as $\eta \to \infty$,

33.12.2 $$\begin{matrix} F_0(\eta,\rho) \\ G_0(\eta,\rho) \end{matrix} \sim \pi^{1/2}(2\eta)^{1/6} \left\{ \begin{matrix} \mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{matrix} \left(1 + \frac{B_1}{\mu} + \frac{B_2}{\mu^2} + \cdots \right) + \begin{matrix} \mathrm{Ai}'(x) \\ \mathrm{Bi}'(x) \end{matrix} \left(\frac{A_1}{\mu} + \frac{A_2}{\mu^2} + \cdots \right) \right\},$$

33.12.3 $$\begin{matrix} F'_0(\eta,\rho) \\ G'_0(\eta,\rho) \end{matrix} \sim -\pi^{1/2}(2\eta)^{-1/6} \left\{ \begin{matrix} \mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{matrix} \left(\frac{B'_1 + xA_1}{\mu} + \frac{B'_2 + xA_2}{\mu^2} + \cdots \right) + \begin{matrix} \mathrm{Ai}'(x) \\ \mathrm{Bi}'(x) \end{matrix} \left(\frac{B_1 + A'_1}{\mu} + \frac{B_2 + A'_2}{\mu^2} + \cdots \right) \right\},$$

uniformly for bounded values of $\left|(\rho - 2\eta)/\eta^{1/3}\right|$. Here Ai and Bi are the Airy functions (§9.2), and

33.12.4 $$A_1 = \tfrac{1}{5}x^2, \quad A_2 = \tfrac{1}{35}(2x^3 + 6), \quad A_3 = \tfrac{1}{15750}(21x^7 + 370x^4 + 580x),$$

33.12.5 $$B_1 = -\tfrac{1}{5}x, \quad B_2 = \tfrac{1}{350}(7x^5 - 30x^2), \quad B_3 = \tfrac{1}{15750}(264x^6 - 290x^3 - 560).$$

In particular,

33.12.6 $$\begin{matrix} F_0(\eta,2\eta) \\ 3^{-1/2}\,G_0(\eta,2\eta) \end{matrix} \sim \frac{\Gamma\left(\frac{1}{3}\right)\omega^{1/2}}{2\sqrt{\pi}} \left(1 \mp \frac{2}{35}\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\frac{1}{\omega^4} - \frac{8}{2025}\frac{1}{\omega^6} \mp \frac{5792}{46\,06875}\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\frac{1}{\omega^{10}} - \cdots \right),$$

33.12.7 $$\begin{matrix} F'_0(\eta,2\eta) \\ 3^{-1/2}\,G'_0(\eta,2\eta) \end{matrix} \sim \frac{\Gamma\left(\frac{2}{3}\right)}{2\sqrt{\pi}\omega^{1/2}} \left(\pm 1 + \frac{1}{15}\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\frac{1}{\omega^2} \pm \frac{2}{14175}\frac{1}{\omega^6} + \frac{1436}{23\,38875}\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\frac{1}{\omega^8} \pm \cdots \right),$$

where $\omega = (\frac{2}{3}\eta)^{1/3}$.

For derivations and additional terms in the expansions in this subsection see Abramowitz and Rabinowitz (1954) and Fröberg (1955).

33.12(ii) Uniform Expansions

With the substitution $\rho = 2\eta z$, Equation (33.2.1) becomes

33.12.8 $$\frac{d^2w}{dz^2} = \left(4\eta^2 \left(\frac{1-z}{z} \right) + \frac{\ell(\ell+1)}{z^2} \right) w.$$

Then, by application of the results given in §§2.8(iii) and 2.8(iv), two sets of asymptotic expansions can be constructed for $F_\ell(\eta,\rho)$ and $G_\ell(\eta,\rho)$ when $\eta \to \infty$.

The first set is in terms of Airy functions and the expansions are uniform for fixed ℓ and $\delta \le z < \infty$, where δ is an arbitrary small positive constant. They would include the results of §33.12(i) as a special case.

The second set is in terms of Bessel functions of orders $2\ell+1$ and $2\ell+2$, and they are uniform for fixed ℓ

and $0 \le z \le 1-\delta$, where δ again denotes an arbitrary small positive constant.

Compare also §33.20(iv).

33.13 Complex Variable and Parameters

The functions $F_\ell(\eta,\rho)$, $G_\ell(\eta,\rho)$, and $H_\ell^{\pm}(\eta,\rho)$ may be extended to noninteger values of ℓ by generalizing $(2\ell+1)! = \Gamma(2\ell+2)$, and supplementing (33.6.5) by a formula derived from (33.2.8) with $U(a,b,z)$ expanded via (13.2.42).

These functions may also be continued analytically to complex values of ρ, η, and ℓ. The quantities $C_\ell(\eta)$, $\sigma_\ell(\eta)$, and R_ℓ, given by (33.2.6), (33.2.10), and (33.4.1), respectively, must be defined consistently so that

33.13.1
$$C_\ell(\eta) = 2^\ell e^{i\sigma_\ell(\eta)-(\pi\eta/2)}\,\Gamma(\ell+1-i\eta)/\,\Gamma(2\ell+2),$$

and

33.13.2
$$R_\ell = (2\ell+1)\,C_\ell(\eta)/\,C_{\ell-1}(\eta).$$

For further information see Dzieciol *et al.* (1999), Thompson and Barnett (1986), and Humblet (1984).

Variables r, ϵ

33.14 Definitions and Basic Properties

33.14(i) Coulomb Wave Equation

Another parametrization of (33.2.1) is given by

33.14.1
$$\frac{d^2w}{dr^2} + \left(\epsilon + \frac{2}{r} - \frac{\ell(\ell+1)}{r^2}\right) w = 0,$$

where

33.14.2
$$r = -\eta\rho, \quad \epsilon = 1/\eta^2.$$

Again, there is a regular singularity at $r=0$ with indices $\ell+1$ and $-\ell$, and an irregular singularity of rank 1 at $r=\infty$. When $\epsilon>0$ the outer turning point is given by

33.14.3
$$r_{\mathrm{tp}}(\epsilon,\ell) = \left(\sqrt{1+\epsilon\ell(\ell+1)}-1\right) \Big/ \epsilon;$$

compare (33.2.2).

33.14(ii) Regular Solution $f(\epsilon,\ell;r)$

The function $f(\epsilon,\ell;r)$ is recessive (§2.7(iii)) at $r=0$, and is defined by

33.14.4
$$f(\epsilon,\ell;r) = \kappa^{\ell+1}\,M_{\kappa,\ell+\frac12}(2r/\kappa)/(2\ell+1)!,$$

or equivalently

33.14.5
$$f(\epsilon,\ell;r) = (2r)^{\ell+1}e^{-r/\kappa}\,M(\ell+1-\kappa,2\ell+2,2r/\kappa)/(2\ell+1)!,$$

where $M_{\kappa,\mu}(z)$ and $M(a,b,z)$ are defined in §§13.14(i) and 13.2(i), and

33.14.6
$$\kappa = \begin{cases} (-\epsilon)^{-1/2}, & \epsilon<0, r>0,\\ -(-\epsilon)^{-1/2}, & \epsilon<0, r<0,\\ \pm i\epsilon^{-1/2}, & \epsilon>0. \end{cases}$$

The choice of sign in the last line of (33.14.6) is immaterial: the same function $f(\epsilon,\ell;r)$ is obtained. This is a consequence of Kummer's transformation (§13.2(vii)).

$f(\epsilon,\ell;r)$ is real and an analytic function of r in the interval $-\infty<r<\infty$, and it is also an analytic function of ϵ when $-\infty<\epsilon<\infty$. This includes $\epsilon=0$, hence $f(\epsilon,\ell;r)$ can be expanded in a convergent power series in ϵ in a neighborhood of $\epsilon=0$ (§33.20(ii)).

33.14(iii) Irregular Solution $h(\epsilon,\ell;r)$

For nonzero values of ϵ and r the function $h(\epsilon,\ell;r)$ is defined by

33.14.7
$$h(\epsilon,\ell;r) = \frac{\Gamma(\ell+1-\kappa)}{\pi\kappa^\ell}\left(W_{\kappa,\ell+\frac12}(2r/\kappa) + (-1)^\ell S(\epsilon,r)\frac{\Gamma(\ell+1+\kappa)}{2(2\ell+1)!}\,M_{\kappa,\ell+\frac12}(2r/\kappa)\right),$$

where κ is given by (33.14.6) and

33.14.8
$$S(\epsilon,r) = \begin{cases} 2\cos\left(\pi|\epsilon|^{-1/2}\right), & \epsilon<0, r>0,\\ 0, & \epsilon<0, r<0,\\ e^{\pi\epsilon^{-1/2}}, & \epsilon>0, r>0,\\ e^{-\pi\epsilon^{-1/2}}, & \epsilon>0, r<0. \end{cases}$$

(Again, the choice of the ambiguous sign in the last line of (33.14.6) is immaterial.)

$h(\epsilon,\ell;r)$ is real and an analytic function of each of r and ϵ in the intervals $-\infty<r<\infty$ and $-\infty<\epsilon<\infty$, except when $r=0$ or $\epsilon=0$.

33.14(iv) Solutions $s(\epsilon,\ell;r)$ and $c(\epsilon,\ell;r)$

The functions $s(\epsilon,\ell;r)$ and $c(\epsilon,\ell;r)$ are defined by

33.14.9
$$\begin{aligned} s(\epsilon,\ell;r) &= (B(\epsilon,\ell)/2)^{1/2}\,f(\epsilon,\ell;r),\\ c(\epsilon,\ell;r) &= (2B(\epsilon,\ell))^{-1/2}\,h(\epsilon,\ell;r), \end{aligned}$$

provided that $\ell<(-\epsilon)^{-1/2}$ when $\epsilon<0$, where

33.14.10
$$B(\epsilon,\ell) = \begin{cases} A(\epsilon,\ell)\left(1-\exp\left(-2\pi/\epsilon^{1/2}\right)\right)^{-1}, & \epsilon>0,\\ A(\epsilon,\ell), & \epsilon\le 0, \end{cases}$$

and

33.14.11
$$A(\epsilon,\ell) = \prod_{k=0}^{\ell}(1+\epsilon k^2).$$

An alternative formula for $A(\epsilon,\ell)$ is

33.14.12
$$A(\epsilon,\ell) = \frac{\Gamma(1+\ell+\kappa)}{\Gamma(\kappa-\ell)}\kappa^{-2\ell-1},$$

the choice of sign in the last line of (33.14.6) again being immaterial.

When $\epsilon < 0$ and $\ell > (-\epsilon)^{-1/2}$ the quantity $A(\epsilon, \ell)$ may be negative, causing $s(\epsilon, \ell; r)$ and $c(\epsilon, \ell; r)$ to become imaginary.

The function $s(\epsilon, \ell; r)$ has the following properties:

33.14.13 $$\int_0^\infty s(\epsilon_1, \ell; r)\, s(\epsilon_2, \ell; r)\, dr = \delta(\epsilon_1 - \epsilon_2),$$

where the right-hand side is the Dirac delta (§1.17). When $\epsilon = -1/n^2$, $n = \ell+1, \ell+2, \dots$, $s(\epsilon, \ell; r)$ is $\exp(-r/n)$ times a polynomial in r, and

33.14.14 $$\phi_{n,\ell}(r) = (-1)^{\ell+1+n}(2/n^3)^{1/2}\, s\big(-1/n^2, \ell; r\big)$$

satisfies

33.14.15 $$\int_0^\infty \phi_{n,\ell}^2(r)\, dr = 1.$$

33.14(v) Wronskians

With arguments ϵ, ℓ, r suppressed,

33.14.16 $$\mathscr{W}\{h, f\} = 2/\pi, \quad \mathscr{W}\{c, s\} = 1/\pi.$$

33.15 Graphics

33.15(i) Line Graphs of the Coulomb Functions $f(\epsilon, \ell; r)$ and $h(\epsilon, \ell; r)$

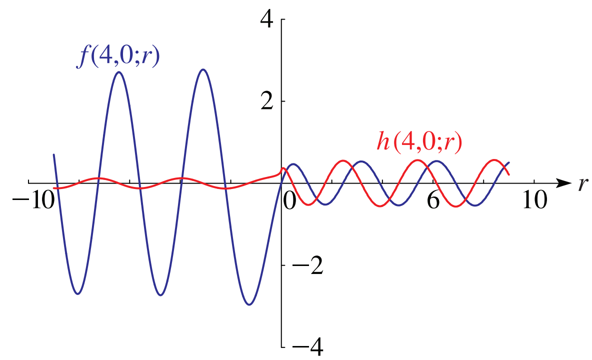

Figure 33.15.1: $f(\epsilon, \ell; r), h(\epsilon, \ell; r)$ with $\ell = 0, \epsilon = 4$.

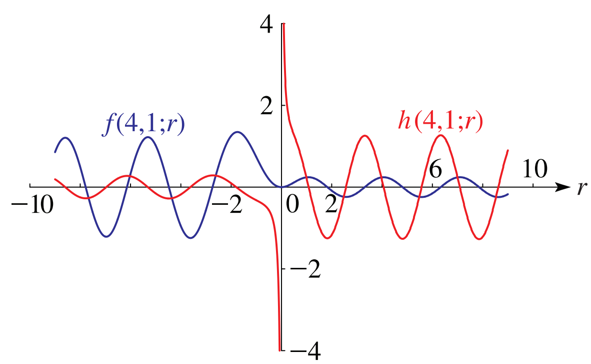

Figure 33.15.2: $f(\epsilon, \ell; r), h(\epsilon, \ell; r)$ with $\ell = 1, \epsilon = 4$.

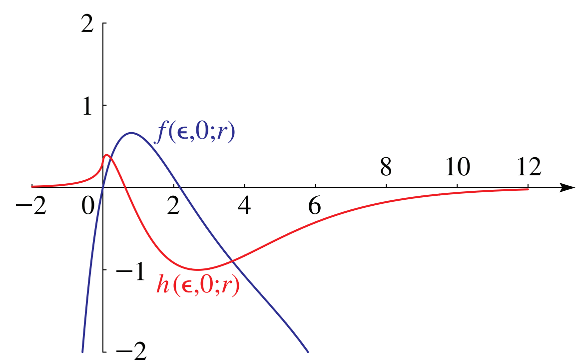

Figure 33.15.3: $f(\epsilon, \ell; r), h(\epsilon, \ell; r)$ with $\ell = 0, \epsilon = -1/\nu^2, \nu = 1.5$.

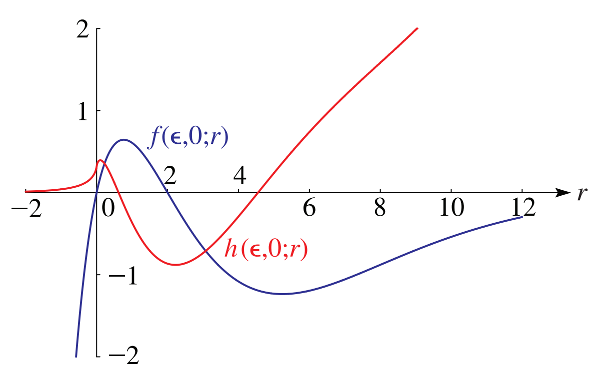

Figure 33.15.4: $f(\epsilon, \ell; r), h(\epsilon, \ell; r)$ with $\ell = 0, \epsilon = -1/\nu^2, \nu = 2$.

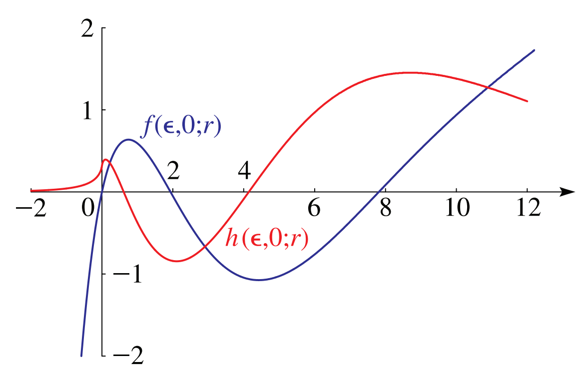

Figure 33.15.5: $f(\epsilon, \ell; r), h(\epsilon, \ell; r)$ with $\ell = 0, \epsilon = -1/\nu^2, \nu = 2.5$.

33.15(ii) Surfaces of the Coulomb Functions $f(\epsilon,\ell;r)$, $h(\epsilon,\ell;r)$, $s(\epsilon,\ell;r)$, and $c(\epsilon,\ell;r)$

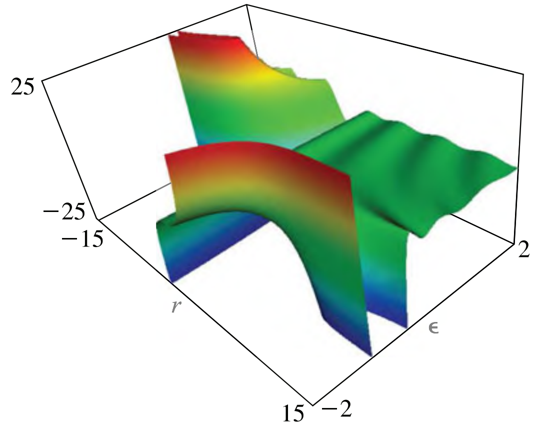

Figure 33.15.6: $f(\epsilon,\ell;r)$ with $\ell=0, -2<\epsilon<2, -15<r<15$.

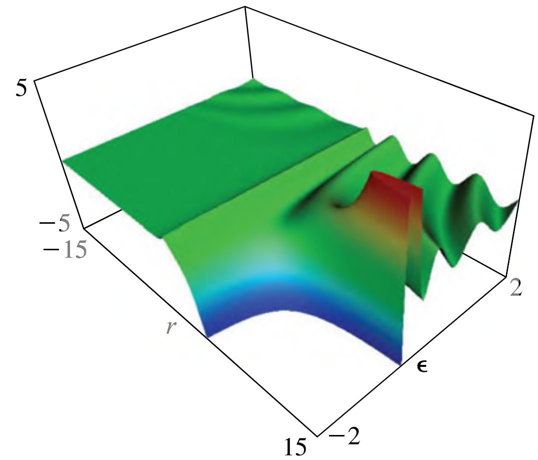

Figure 33.15.7: $h(\epsilon,\ell;r)$ with $\ell=0, -2<\epsilon<2, -15<r<15$.

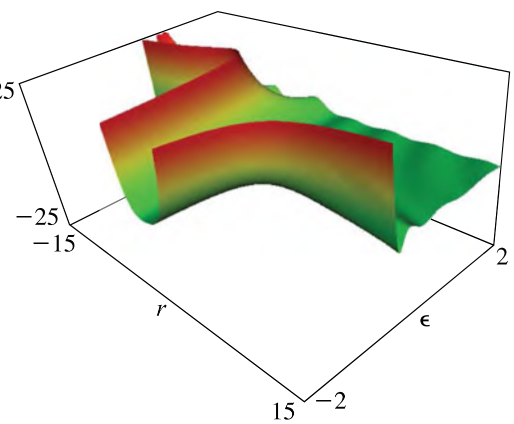

Figure 33.15.8: $f(\epsilon,\ell;r)$ with $\ell=1, -2<\epsilon<2, -15<r<15$.

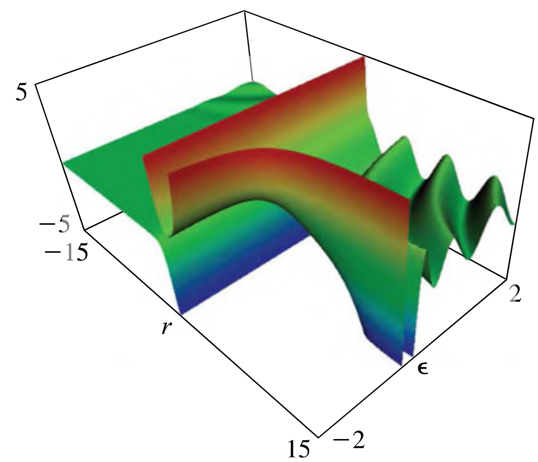

Figure 33.15.9: $h(\epsilon,\ell;r)$ with $\ell=1, -2<\epsilon<2, -15<r<15$.

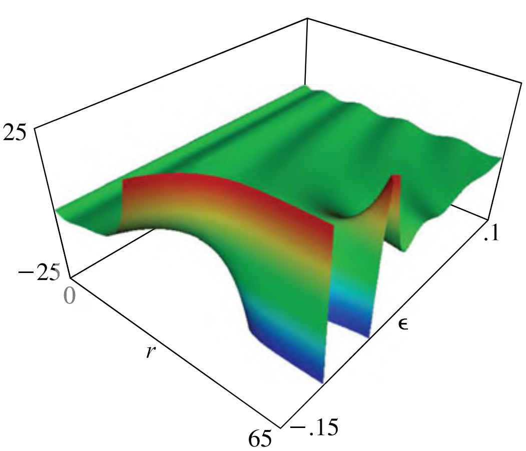

Figure 33.15.10: $s(\epsilon,\ell;r)$ with $\ell=0, -0.15<\epsilon<0.10, 0<r<65$.

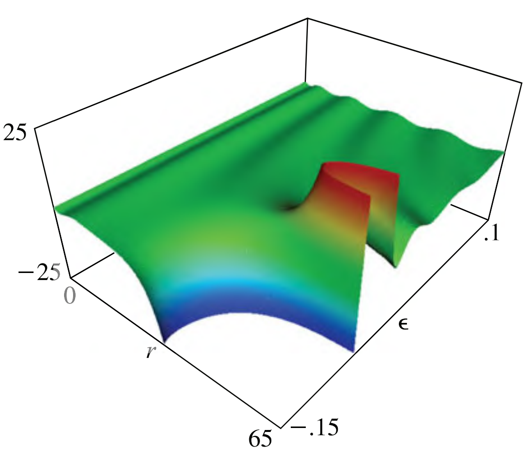

Figure 33.15.11: $c(\epsilon,\ell;r)$ with $\ell=0, -0.15<\epsilon<0.10, 0<r<65$.

33.16 Connection Formulas

33.16(i) F_ℓ and G_ℓ in Terms of f and h

33.16.1 $$F_\ell(\eta,\rho)=\frac{(2\ell+1)!\,C_\ell(\eta)}{(-2\eta)^{\ell+1}}\,f\left(1/\eta^2,\ell;-\eta\rho\right),$$

33.16.2 $$G_\ell(\eta,\rho)=\frac{\pi(-2\eta)^\ell}{(2\ell+1)!\,C_\ell(\eta)}\,h\left(1/\eta^2,\ell;-\eta\rho\right),$$

where $C_\ell(\eta)$ is given by (33.2.5) or (33.2.6).

33.16(ii) f and h in Terms of F_ℓ and G_ℓ when $\epsilon>0$

When $\epsilon>0$ denote

33.16.3 $$\tau=\epsilon^{1/2}(>0),$$

and again define $A(\epsilon,\ell)$ by (33.14.11) or (33.14.12). Then for $r>0$

33.16.4 $$f(\epsilon,\ell;r)=\left(\frac{2}{\pi\tau}\frac{1-e^{-2\pi/\tau}}{A(\epsilon,\ell)}\right)^{1/2}F_\ell(-1/\tau,\tau r),$$

33.16.5 $$h(\epsilon,\ell;r)=\left(\frac{2}{\pi\tau}\frac{A(\epsilon,\ell)}{1-e^{-2\pi/\tau}}\right)^{1/2}G_\ell(-1/\tau,\tau r).$$

Alternatively, for $r<0$

33.16.6 $$f(\epsilon,\ell;r)=(-1)^{\ell+1}\left(\frac{2}{\pi\tau}\frac{e^{2\pi/\tau}-1}{A(\epsilon,\ell)}\right)^{1/2}F_\ell(1/\tau,-\tau r),$$

33.16.7 $$h(\epsilon,\ell;r)=(-1)^{\ell}\left(\frac{2}{\pi\tau}\frac{A(\epsilon,\ell)}{e^{2\pi/\tau}-1}\right)^{1/2}G_\ell(1/\tau,-\tau r).$$

33.16(iii) f and h in Terms of $W_{\kappa,\mu}(z)$ when $\epsilon<0$

When $\epsilon<0$ denote

33.16.8 $$\nu=1/(-\epsilon)^{1/2}(>0),$$

33.16.9 $$\zeta_\ell(\nu,r)=W_{\nu,\ell+\frac12}(2r/\nu),\qquad \xi_\ell(\nu,r)=\Re\left(e^{i\pi\nu}\,W_{-\nu,\ell+\frac12}\left(e^{i\pi}2r/\nu\right)\right),$$

and again define $A(\epsilon,\ell)$ by (33.14.11) or (33.14.12). Then for $r>0$

33.16.10 $$f(\epsilon,\ell;r)=(-1)^\ell\nu^{\ell+1}\left(-\frac{\cos(\pi\nu)\zeta_\ell(\nu,r)}{\Gamma(\ell+1+\nu)}+\frac{\sin(\pi\nu)\,\Gamma(\nu-\ell)\xi_\ell(\nu,r)}{\pi}\right),$$

33.16.11 $$h(\epsilon,\ell;r)=(-1)^\ell\nu^{\ell+1}A(\epsilon,\ell)\left(\frac{\sin(\pi\nu)\zeta_\ell(\nu,r)}{\Gamma(\ell+1+\nu)}+\frac{\cos(\pi\nu)\,\Gamma(\nu-\ell)\xi_\ell(\nu,r)}{\pi}\right).$$

Alternatively, for $r<0$

33.16.12 $$f(\epsilon,\ell;r)=\frac{(-1)^\ell\nu^{\ell+1}}{\pi}\left(-\frac{\pi\xi_\ell(-\nu,r)}{\Gamma(\ell+1+\nu)}+\sin(\pi\nu)\cos(\pi\nu)\,\Gamma(\nu-\ell)\zeta_\ell(-\nu,r)\right),$$

33.16.13 $$h(\epsilon,\ell;r)=(-1)^\ell\nu^{\ell+1}A(\epsilon,\ell)\,\Gamma(\nu-\ell)\zeta_\ell(-\nu,r)/\pi.$$

33.16(iv) s and c in Terms of F_ℓ and G_ℓ when $\epsilon>0$

When $\epsilon>0$, again denote τ by (33.16.3). Then for $r>0$

33.16.14 $$s(\epsilon,\ell;r)=(\pi\tau)^{-1/2}\,F_\ell(-1/\tau,\tau r),\qquad c(\epsilon,\ell;r)=(\pi\tau)^{-1/2}\,G_\ell(-1/\tau,\tau r).$$

Alternatively, for $r<0$

33.16.15 $$s(\epsilon,\ell;r)=(\pi\tau)^{-1/2}\,F_\ell(1/\tau,-\tau r),\qquad c(\epsilon,\ell;r)=(\pi\tau)^{-1/2}\,G_\ell(1/\tau,-\tau r).$$

33.16(v) s and c in Terms of $W_{\kappa,\mu}(z)$ when $\epsilon<0$

When $\epsilon<0$ denote ν, $\zeta_\ell(\nu,r)$, and $\xi_\ell(\nu,r)$ by (33.16.8) and (33.16.9). Also denote

33.16.16 $$K(\nu,\ell)=\left(\nu^2\,\Gamma(\nu+\ell+1)\,\Gamma(\nu-\ell)\right)^{-1/2}.$$

Then for $r>0$

33.16.17 $$s(\epsilon,\ell;r)=\frac{(-1)^\ell}{2\nu^{1/2}}\left(\frac{\sin(\pi\nu)}{\pi K(\nu,\ell)}\xi_\ell(\nu,r)-\cos(\pi\nu)\nu^2K(\nu,\ell)\zeta_\ell(\nu,r)\right),$$
$$c(\epsilon,\ell;r)=\frac{(-1)^\ell}{2\nu^{1/2}}\left(\frac{\cos(\pi\nu)}{\pi K(\nu,\ell)}\xi_\ell(\nu,r)+\sin(\pi\nu)\nu^2K(\nu,\ell)\zeta_\ell(\nu,r)\right).$$

Alternatively, for $r<0$

33.16.18 $$s(\epsilon,\ell;r)=\frac{(-1)^{\ell+1}}{2^{1/2}}\left(\frac{\nu^{3/2}}{K(\nu,\ell)}\xi_\ell(-\nu,r)-\frac{\sin(\pi\nu)\cos(\pi\nu)}{\pi\nu^{1/2}}K(\nu,\ell)\zeta_\ell(-\nu,r)\right),$$
$$c(\epsilon,\ell;r)=\frac{(-1)^\ell}{\pi(2\nu)^{1/2}}K(\nu,\ell)\zeta_\ell(-\nu,r).$$

33.17 Recurrence Relations and Derivatives

33.17.1 $$(\ell+1)r\,f(\epsilon,\ell-1;r)-(2\ell+1)\left(\ell(\ell+1)-r\right)f(\epsilon,\ell;r)+\ell\left(1+(\ell+1)^2\epsilon\right)r\,f(\epsilon,\ell+1;r)=0,$$

33.17.2 $$(\ell+1)\left(1+\ell^2\epsilon\right)r\,h(\epsilon,\ell-1;r)-(2\ell+1)\left(\ell(\ell+1)-r\right)h(\epsilon,\ell;r)+\ell r\,h(\epsilon,\ell+1;r)=0,$$

33.17.3 $$(\ell+1)r\,f'(\epsilon,\ell;r)=\left((\ell+1)^2-r\right)f(\epsilon,\ell;r)-\left(1+(\ell+1)^2\epsilon\right)r\,f(\epsilon,\ell+1;r),$$

33.17.4 $$(\ell+1)r\,h'(\epsilon,\ell;r)=\left((\ell+1)^2-r\right)h(\epsilon,\ell;r)-r\,h(\epsilon,\ell+1;r).$$

33.18 Limiting Forms for Large ℓ

As $\ell\to\infty$ with ϵ and r ($\neq 0$) fixed,

33.18.1 $$f(\epsilon,\ell;r)\sim\frac{(2r)^{\ell+1}}{(2\ell+1)!},\quad h(\epsilon,\ell;r)\sim\frac{(2\ell)!}{\pi(2r)^\ell}.$$

33.19 Power-Series Expansions in r

33.19.1 $$f(\epsilon,\ell;r)=r^{\ell+1}\sum_{k=0}^{\infty}\alpha_k r^k,$$

where

33.19.2
$$\alpha_0=2^{\ell+1}/(2\ell+1)!,\quad \alpha_1=-\alpha_0/(\ell+1),$$
$$k(k+2\ell+1)\alpha_k+2\alpha_{k-1}+\epsilon\alpha_{k-2}=0,\quad k=2,3,\dots.$$

33.19.3
$$2\pi\,h(\epsilon,\ell;r)=\sum_{k=0}^{2\ell}\frac{(2\ell-k)!\gamma_k}{k!}(2r)^{k-\ell}-\sum_{k=0}^{\infty}\delta_k r^{k+\ell+1}-A(\epsilon,\ell)\left(2\ln|2r/\kappa|+\Re\psi(\ell+1+\kappa)+\Re\psi(-\ell+\kappa)\right)f(\epsilon,\ell;r),\quad r\neq 0.$$

Here κ is defined by (33.14.6), $A(\epsilon,\ell)$ is defined by (33.14.11) or (33.14.12), $\gamma_0=1$, $\gamma_1=1$, and

33.19.4
$$\gamma_k-\gamma_{k-1}+\tfrac{1}{4}(k-1)(k-2\ell-2)\epsilon\gamma_{k-2}=0,\quad k=2,3,\dots.$$

Also,

33.19.5
$$\delta_0=\left(\beta_{2\ell+1}-2(\psi(2\ell+2)+\psi(1))A(\epsilon,\ell)\right)\alpha_0,$$
$$\delta_1=\left(\beta_{2\ell+2}-2(\psi(2\ell+3)+\psi(2))A(\epsilon,\ell)\right)\alpha_1,$$

33.19.6
$$k(k+2\ell+1)\delta_k+2\delta_{k-1}+\epsilon\delta_{k-2}+2(2k+2\ell+1)A(\epsilon,\ell)\alpha_k=0,\quad k=2,3,\dots,$$

with $\beta_0=\beta_1=0$, and

33.19.7
$$\beta_k-\beta_{k-1}+\tfrac{1}{4}(k-1)(k-2\ell-2)\epsilon\beta_{k-2}+\tfrac{1}{2}(k-1)\epsilon\gamma_{k-2}=0,\quad k=2,3,\dots.$$

The expansions (33.19.1) and (33.19.3) converge for all finite values of r, except $r=0$ in the case of (33.19.3).

33.20 Expansions for Small $|\epsilon|$

33.20(i) Case $\epsilon=0$

33.20.1
$$f(0,\ell;r)=(2r)^{1/2}J_{2\ell+1}\left(\sqrt{8r}\right),$$
$$h(0,\ell;r)=-(2r)^{1/2}Y_{2\ell+1}\left(\sqrt{8r}\right),\quad r>0,$$

33.20.2
$$f(0,\ell;r)=(-1)^{\ell+1}(2|r|)^{1/2}I_{2\ell+1}\left(\sqrt{8|r|}\right),$$
$$h(0,\ell;r)=(-1)^{\ell}(2/\pi)(2|r|)^{1/2}K_{2\ell+1}\left(\sqrt{8|r|}\right),\quad r<0.$$

For the functions J, Y, I, and K see §§10.2(ii), 10.25(ii).

33.20(ii) Power-Series in ϵ for the Regular Solution

33.20.3 $$f(\epsilon,\ell;r)=\sum_{k=0}^{\infty}\epsilon^k\mathsf{F}_k(\ell;r),$$

where

33.20.4
$$\mathsf{F}_k(\ell;r)=\sum_{p=2k}^{3k}(2r)^{(p+1)/2}C_{k,p}\,J_{2\ell+1+p}\left(\sqrt{8r}\right),\quad r>0,$$

33.20.5
$$\mathsf{F}_k(\ell;r)=\sum_{p=2k}^{3k}(-1)^{\ell+1+p}(2|r|)^{(p+1)/2}C_{k,p}\,I_{2\ell+1+p}\left(\sqrt{8|r|}\right),\quad r<0.$$

The functions J and I are as in §§10.2(ii), 10.25(ii), and the coefficients $C_{k,p}$ are given by $C_{0,0}=1$, $C_{1,0}=0$, and

33.20.6
$$C_{k,p}=0,\quad p<2k \text{ or } p>3k,$$
$$C_{k,p}=\left(-(2\ell+p)C_{k-1,p-2}+C_{k-1,p-3}\right)/(4p),\quad k>0,\ 2k\le p\le 3k.$$

The series (33.20.3) converges for all r and ϵ.

33.20(iii) Asymptotic Expansion for the Irregular Solution

As $\epsilon \to 0$ with ℓ and r fixed,

33.20.7 $$h(\epsilon, \ell; r) \sim -A(\epsilon, \ell) \sum_{k=0}^{\infty} \epsilon^k \mathsf{H}_k(\ell; r),$$

where $A(\epsilon, \ell)$ is given by (33.14.11), (33.14.12), and

33.20.8 $$\mathsf{H}_k(\ell; r) = \sum_{p=2k}^{3k} (2r)^{(p+1)/2} C_{k,p}\, Y_{2\ell+1+p}\left(\sqrt{8r}\right), \quad r > 0,$$

33.20.9 $$\mathsf{H}_k(\ell; r) = (-1)^{\ell+1} \frac{2}{\pi} \sum_{p=2k}^{3k} (2|r|)^{(p+1)/2} C_{k,p}\, K_{2\ell+1+p}\left(\sqrt{8|r|}\right), \quad r < 0.$$

The functions Y and K are as in §§10.2(ii), 10.25(ii), and the coefficients $C_{k,p}$ are given by (33.20.6).

33.20(iv) Uniform Asymptotic Expansions

For a comprehensive collection of asymptotic expansions that cover $f(\epsilon, \ell; r)$ and $h(\epsilon, \ell; r)$ as $\epsilon \to 0\pm$ and are uniform in r, including unbounded values, see Curtis (1964a, §7). These expansions are in terms of elementary functions, Airy functions, and Bessel functions of orders $2\ell + 1$ and $2\ell + 2$.

33.21 Asymptotic Approximations for Large $|r|$

33.21(i) Limiting Forms

We indicate here how to obtain the limiting forms of $f(\epsilon, \ell; r)$, $h(\epsilon, \ell; r)$, $s(\epsilon, \ell; r)$, and $c(\epsilon, \ell; r)$ as $r \to \pm\infty$, with ϵ and ℓ fixed, in the following cases:

(a) When $r \to \pm\infty$ with $\epsilon > 0$, Equations (33.16.4)–(33.16.7) are combined with (33.10.1).

(b) When $r \to \pm\infty$ with $\epsilon < 0$, Equations (33.16.10)–(33.16.13) are combined with

33.21.1 $$\zeta_\ell(\nu, r) \sim e^{-r/\nu}(2r/\nu)^{\nu}, \quad \xi_\ell(\nu, r) \sim e^{r/\nu}(2r/\nu)^{-\nu}, \qquad r \to \infty,$$

33.21.2 $$\zeta_\ell(-\nu, r) \sim e^{r/\nu}(-2r/\nu)^{-\nu}, \quad \xi_\ell(-\nu, r) \sim e^{-r/\nu}(-2r/\nu)^{\nu}, \qquad r \to -\infty.$$

Corresponding approximations for $s(\epsilon, \ell; r)$ and $c(\epsilon, \ell; r)$ as $r \to \infty$ can be obtained via (33.16.17), and as $r \to -\infty$ via (33.16.18).

(c) When $r \to \pm\infty$ with $\epsilon = 0$, combine (33.20.1), (33.20.2) with §§10.7(ii), 10.30(ii).

33.21(ii) Asymptotic Expansions

For asymptotic expansions of $f(\epsilon, \ell; r)$ and $h(\epsilon, \ell; r)$ as $r \to \pm\infty$ with ϵ and ℓ fixed, see Curtis (1964a, §6).

Physical Applications

33.22 Particle Scattering and Atomic and Molecular Spectra

33.22(i) Schrödinger Equation

With e denoting here the elementary charge, the Coulomb potential between two point particles with charges $Z_1 e, Z_2 e$ and masses m_1, m_2 separated by a distance s is $V(s) = Z_1 Z_2 e^2/(4\pi\epsilon_0 s) = Z_1 Z_2 \alpha\hbar c/s$, where Z_j are atomic numbers, ϵ_0 is the electric constant, α is the fine structure constant, and $\hbar$ is the reduced Planck's constant. The reduced mass is $m = m_1 m_2/(m_1 + m_2)$, and at energy of relative motion E with relative orbital angular momentum $\ell\hbar$, the *Schrödinger equation* for the radial wave function $w(s)$ is given by

33.22.1 $$\left(-\frac{\hbar^2}{2m}\left(\frac{d^2}{ds^2} - \frac{\ell(\ell+1)}{s^2}\right) + \frac{Z_1 Z_2 \alpha\hbar c}{s}\right) w = Ew,$$

With the substitutions

33.22.2 $$\mathsf{k} = (2mE/\hbar^2)^{1/2}, \quad Z = mZ_1 Z_2 \alpha c/\hbar, \quad x = s,$$

(33.22.1) becomes

33.22.3 $$\frac{d^2 w}{dx^2} + \left(\mathsf{k}^2 - \frac{2Z}{x} - \frac{\ell(\ell+1)}{x^2}\right) w = 0.$$

33.22(ii) Definitions of Variables

k Scaling

The k-scaled variables ρ and η of §33.2 are given by

33.22.4 $$\rho = s(2mE/\hbar^2)^{1/2}, \quad \eta = Z_1 Z_2 \alpha c (m/(2E))^{1/2}.$$

At positive energies $E > 0$, $\rho \geq 0$, and:

Attractive potentials:	$Z_1 Z_2 < 0$, $\eta < 0$.
Zero potential ($V = 0$):	$Z_1 Z_2 = 0$, $\eta = 0$.
Repulsive potentials:	$Z_1 Z_2 > 0$, $\eta > 0$.

Positive-energy functions correspond to processes such as Rutherford scattering and Coulomb excitation of nuclei (Alder *et al.* (1956)), and atomic photo-ionization and electron-ion collisions (Bethe and Salpeter (1977)).

At negative energies $E < 0$ and both ρ and η are purely imaginary. The negative-energy functions are widely used in the description of atomic and molecular spectra; see Bethe and Salpeter (1977), Seaton (1983), and Aymar *et al.* (1996). In these applications, the Z-scaled variables r and ϵ are more convenient.

Z Scaling

The Z-scaled variables r and ϵ of §33.14 are given by

33.22.5 $$r = -Z_1 Z_2 (mc\alpha/\hbar)s, \quad \epsilon = E/(Z_1^2 Z_2^2 mc^2 \alpha^2/2).$$

For $Z_1 Z_2 = -1$ and $m = m_e$, the electron mass, the scaling factors in (33.22.5) reduce to the Bohr radius, $a_0 = \hbar/(m_e c\alpha)$, and to a multiple of the Rydberg constant,

$R_\infty = m_e c\alpha^2/(2\hbar)$.

Attractive potentials: $Z_1 Z_2 < 0$, $r > 0$.
Zero potential ($V = 0$): $Z_1 Z_2 = 0$, $r = 0$.
Repulsive potentials: $Z_1 Z_2 > 0$, $r < 0$.

$i\mathsf{k}$ Scaling

The $i\mathsf{k}$-scaled variables z and κ of §13.2 are given by

33.22.6 $$z = 2is(2mE/\hbar^2)^{1/2}, \quad \kappa = iZ_1 Z_2 \alpha c(m/(2E))^{1/2}.$$

Attractive potentials: $Z_1 Z_2 < 0$, $\Im\kappa < 0$.
Zero potential ($V = 0$): $Z_1 Z_2 = 0$, $\kappa = 0$.
Repulsive potentials: $Z_1 Z_2 > 0$, $\Im\kappa > 0$.

Customary variables are (ϵ, r) in atomic physics and (η, ρ) in atomic and nuclear physics. Both variable sets may be used for attractive and repulsive potentials: the (ϵ, r) set cannot be used for a zero potential because this would imply $r = 0$ for all s, and the (η, ρ) set cannot be used for zero energy E because this would imply $\rho = 0$ always.

33.22(iii) Conversions Between Variables

33.22.7 $$r = -\eta\rho, \quad \epsilon = 1/\eta^2, \qquad Z \text{ from } \mathsf{k}.$$

33.22.8 $$z = 2i\rho, \quad \kappa = i\eta, \qquad i\mathsf{k} \text{ from } \mathsf{k}.$$

33.22.9 $$\rho = z/(2i), \quad \eta = \kappa/i, \qquad \mathsf{k} \text{ from } i\mathsf{k}.$$

33.22.10 $$r = \kappa z/2, \quad \epsilon = -1/\kappa^2, \qquad Z \text{ from } i\mathsf{k}.$$

33.22.11 $$\eta = \pm\epsilon^{-1/2}, \quad \rho = -r/\eta, \qquad \mathsf{k} \text{ from } Z.$$

33.22.12 $$\kappa = \pm(-\epsilon)^{-1/2}, \quad z = 2r/\kappa, \qquad i\mathsf{k} \text{ from } Z.$$

Resolution of the ambiguous signs in (33.22.11), (33.22.12) depends on the sign of Z/k in (33.22.3). See also §§33.14(ii), 33.14(iii), 33.22(i), and 33.22(ii).

33.22(iv) Klein–Gordon and Dirac Equations

The relativistic motion of spinless particles in a Coulomb field, as encountered in pionic atoms and pion-nucleon scattering (Backenstoss (1970)) is described by a Klein–Gordon equation equivalent to (33.2.1); see Barnett (1981a). The motion of a relativistic electron in a Coulomb field, which arises in the theory of the electronic structure of heavy elements (Johnson (2007)), is described by a Dirac equation. The solutions to this equation are closely related to the Coulomb functions; see Greiner *et al.* (1985).

33.22(v) Asymptotic Solutions

The Coulomb solutions of the Schrödinger and Klein–Gordon equations are almost always used in the external region, outside the range of any non-Coulomb forces or couplings.

For scattering problems, the interior solution is then matched to a linear combination of a pair of Coulomb functions, $F_\ell(\eta, \rho)$ and $G_\ell(\eta, \rho)$, or $f(\epsilon, \ell; r)$ and $h(\epsilon, \ell; r)$, to determine the scattering S-matrix and also the correct normalization of the interior wave solutions; see Bloch *et al.* (1951).

For bound-state problems only the exponentially decaying solution is required, usually taken to be the Whittaker function $W_{-\eta,\ell+\frac{1}{2}}(2\rho)$. The functions $\phi_{n,\ell}(r)$ defined by (33.14.14) are the hydrogenic bound states in attractive Coulomb potentials; their polynomial components are often called *associated Laguerre functions*; see Christy and Duck (1961) and Bethe and Salpeter (1977).

33.22(vi) Solutions Inside the Turning Point

The penetrability of repulsive Coulomb potential barriers is normally expressed in terms of the quantity $\rho/(F_\ell^2(\eta, \rho) + G_\ell^2(\eta, \rho))$ (Mott and Massey (1956, pp. 63–65)). The WKBJ approximations of §33.23(vii) may also be used to estimate the penetrability.

33.22(vii) Complex Variables and Parameters

The Coulomb functions given in this chapter are most commonly evaluated for real values of ρ, r, η, ϵ and nonnegative integer values of ℓ, but they may be continued analytically to complex arguments and order ℓ as indicated in §33.13.

Examples of applications to noninteger and/or complex variables are as follows.

- Scattering at complex energies. See for example McDonald and Nuttall (1969).
- Searches for resonances as poles of the S-matrix in the complex half-plane $\Im\mathsf{k} < 0$. See for example Csótó and Hale (1997).
- Regge poles at complex values of ℓ. See for example Takemasa *et al.* (1979).
- Eigenstates using complex-rotated coordinates $r \to re^{i\theta}$, so that resonances have square-integrable eigenfunctions. See for example Halley *et al.* (1993).
- Solution of relativistic Coulomb equations. See for example Cooper *et al.* (1979) and Barnett (1981b).

- Gravitational radiation. See for example Berti and Cardoso (2006).

For further examples see Humblet (1984).

Computation

33.23 Methods of Computation

33.23(i) Methods for the Confluent Hypergeometric Functions

The methods used for computing the Coulomb functions described below are similar to those in §13.29.

33.23(ii) Series Solutions

The power-series expansions of §§33.6 and 33.19 converge for all finite values of the radii ρ and r, respectively, and may be used to compute the regular and irregular solutions. Cancellation errors increase with increases in ρ and $|r|$, and may be estimated by comparing the final sum of the series with the largest partial sum. Use of extended-precision arithmetic increases the radial range that yields accurate results, but eventually other methods must be employed, for example, the asymptotic expansions of §§33.11 and 33.21.

33.23(iii) Integration of Defining Differential Equations

When numerical values of the Coulomb functions are available for some radii, their values for other radii may be obtained by direct numerical integration of equations (33.2.1) or (33.14.1), provided that the integration is carried out in a stable direction (§3.7). Thus the regular solutions can be computed from the power-series expansions (§§33.6, 33.19) for small values of the radii and then integrated in the direction of increasing values of the radii. On the other hand, the irregular solutions of §§33.2(iii) and 33.14(iii) need to be integrated in the direction of decreasing radii beginning, for example, with values obtained from asymptotic expansions (§§33.11 and 33.21).

33.23(iv) Recurrence Relations

In a similar manner to §33.23(iii) the recurrence relations of §§33.4 or 33.17 can be used for a range of values of the integer ℓ, provided that the recurrence is carried out in a stable direction (§3.6). This implies decreasing ℓ for the regular solutions and increasing ℓ for the irregular solutions of §§33.2(iii) and 33.14(iii).

33.23(v) Continued Fractions

§33.8 supplies continued fractions for F'_ℓ / F_ℓ and $H^{\pm\prime}_\ell / H^{\pm}_\ell$. Combined with the Wronskians (33.2.12), the values of F_ℓ, G_ℓ, and their derivatives can be extracted. Inside the turning points, that is, when $\rho < \rho_{\mathrm{tp}}(\eta,\ell)$, there can be a loss of precision by a factor of approximately $|G_\ell|^2$.

33.23(vi) Other Numerical Methods

Curtis (1964a, §10) describes the use of series, radial integration, and other methods to generate the tables listed in §33.24.

Bardin *et al.* (1972) describes ten different methods for the calculation of F_ℓ and G_ℓ, valid in different regions of the (η,ρ)-plane.

Thompson and Barnett (1985, 1986) and Thompson (2004) use combinations of series, continued fractions, and Padé-accelerated asymptotic expansions (§3.11(iv)) for the analytic continuations of Coulomb functions.

Noble (2004) obtains double-precision accuracy for $W_{-\eta,\mu}(2\rho)$ for a wide range of parameters using a combination of recurrence techniques, power-series expansions, and numerical quadrature; compare (33.2.7).

33.23(vii) WKBJ Approximations

WKBJ approximations (§2.7(iii)) for $\rho > \rho_{\mathrm{tp}}(\eta,\ell)$ are presented in Hull and Breit (1959) and Seaton and Peach (1962: in Eq. (12) $(\rho-c)/c$ should be $(\rho-c)/\rho$). A set of consistent second-order WKBJ formulas is given by Burgess (1963: in Eq. (16) $3\kappa^2+2$ should be $3\kappa^2c+2$). Seaton (1984) estimates the accuracies of these approximations.

Hull and Breit (1959) and Barnett (1981b) give WKBJ approximations for F_0 and G_0 in the region inside the turning point: $\rho < \rho_{\mathrm{tp}}(\eta,\ell)$.

33.24 Tables

- Abramowitz and Stegun (1964, Chapter 14) tabulates $F_0(\eta,\rho)$, $G_0(\eta,\rho)$, $F'_0(\eta,\rho)$, and $G'_0(\eta,\rho)$ for $\eta = 0.5(.5)20$ and $\rho = 1(1)20$, 5S; $C_0(\eta)$ for $\eta = 0(.05)3$, 6S.

- Curtis (1964a) tabulates $P_\ell(\epsilon,r)$, $Q_\ell(\epsilon,r)$ (§33.1), and related functions for $\ell = 0,1,2$ and $\epsilon = -2(.2)2$, with $x = 0(.1)4$ for $\epsilon < 0$ and $x = 0(.1)10$ for $\epsilon \geq 0$; 6D.

For earlier tables see Hull and Breit (1959) and Fletcher *et al.* (1962, §22.59).

33.25 Approximations

Cody and Hillstrom (1970) provides rational approximations of the phase shift $\sigma_0(\eta) = \operatorname{ph}\Gamma(1+i\eta)$ (see (33.2.10)) for the ranges $0 \le \eta \le 2$, $2 \le \eta \le 4$, and $4 \le \eta \le \infty$. Maximum relative errors range from 1.09×10^{-20} to 4.24×10^{-19}.

33.26 Software

See http://dlmf.nist.gov/33.26.

References

General References

The main references used in writing this chapter are Hull and Breit (1959), Thompson and Barnett (1986), and Seaton (2002). For additional bibliographic reading see also the General References in Chapter 13.

Sources

The following list gives the references or other indications of proofs that were used in constructing the various sections of this chapter. These sources supplement the references that are quoted in the text.

§33.2 Yost *et al.* (1936), Hull and Breit (1959, pp. 409–410).

§33.3 These graphics were produced at NIST.

§33.4 Powell (1947).

§33.5 Yost *et al.* (1936), Hull and Breit (1959, pp. 435–436), Wheeler (1937), Biedenharn *et al.* (1955). For (33.5.9) combine the second formula in (5.4.2) with (5.11.7).

§33.6 For (33.6.5) use the definition (33.2.8) with $U(a,b,z)$ expanded as in (13.2.9). For (33.6.4) use (33.2.4) with Eq. (1.12) of Buchholz (1969).

§33.7 Hull and Breit (1959, pp. 413–416). For (33.7.1) see also Lowan and Horenstein (1942), with change of variable $\xi = 1 - t$ in the integral that follows Eq. (8). For (33.7.2) see also Hoisington and Breit (1938). For (33.7.3) see also Bloch *et al.* (1950). For (33.7.4) see also Newton (1952).

§33.9 The convergence of (33.9.1) follows from the asymptotic forms, for large k, of a_k (obtained by application of §2.9(i)) and $\mathsf{j}_{\ell+k}(\rho)$ (obtained from (10.19.1) and (10.47.3)). For (33.9.3) see Yost *et al.* (1936), Abramowitz (1954), and Humblet (1985). For (33.9.4) see Curtis (1964a, §5.1). For (33.9.6) see Yost *et al.* (1936) and Abramowitz (1954).

§33.10 Yost *et al.* (1936), Fröberg (1955), Humblet (1984), Humblet (1985, Eqs. 2.10a,b and 4.7a,b). For (33.10.6) and (33.10.10) use (33.2.5), (33.2.10), and §5.11(i).

§33.11 Fröberg (1955).

§33.14 Curtis (1964a, pp. ix–xxv), Seaton (1983), Seaton (2002, Eqs. 3, 4, 7, 9, 14, 22, 47, 49, 51, 109, 113–116, 122–125, 131, and §2.3). For (33.14.11) and (33.14.12) see Humblet (1985, Eqs. 1.4a,b), Seaton (1982, Eq. 2.4.4).

§33.15 These graphics were produced at NIST.

§33.16 Seaton (2002, Eqs. 104–109, 119–121, 130, 131). (33.16.3)–(33.16.7) are generalizations of Seaton (2002, Eqs. 88, 90, 93, 95). For (33.16.14) and (33.16.15) combine (33.14.9) with (33.16.4)–(33.16.7). For (33.16.17) and (33.16.18) combine (33.14.6), (33.14.9)–(33.14.12), (33.16.10)–(33.16.13), and (33.16.16).

§33.17 Seaton (2002, Eqs. 77, 78, 82).

§33.18 Combine (33.5.8) and (33.16.1), (33.16.2). For $f(\epsilon,\ell;r)$ (33.19.1) can also be used.

§33.19 Seaton (2002, Eqs. 15–17, 31–48).

§33.20 Seaton (2002, Eqs. 58, 59, 64, 67–70, 96, 98, 100, 102 (corrected)).

§33.21 Seaton (2002, Eqs. 104, 107), or apply (13.14.21) to (33.16.9).

§33.23 Stable integration directions for the differential equations are determined by comparison of the asymptotic behavior of the solutions as the radii tend to infinity and also as the radii tend to zero (§§33.11, 33.21; §§33.6, 33.19). Stable recurrence directions for §33.4 are determined by the asymptotic form of $F_\ell(\eta,\rho)/G_\ell(\eta,\rho)$ as $\ell \to \infty$; see (33.5.8) and (33.5.9). For §33.17 see §33.18.